- $t(\hat{a}) = e^{i\frac{\vec{k}'\cdot\vec{a}'}{\hbar}} = e^{i\vec{k}'\cdot\vec{a}'}$

$$t(\hat{a})\cdot t(\hat{b}) = e^{i\vec{k}'\cdot\vec{a}'}\cdot e^{i\vec{k}'\cdot\vec{b}'}$$

$$e^{A}\,e^{B} = e^{(A+B)}\,e^{-\frac{1}{2}[A,B]\lambda}$$

$$\Rightarrow\quad e^{\lambda A}\,e^{\lambda B} = e^{\lambda(A+B)}\cancel{-t\lambda^2}\,e^{t\lambda^2[A,B]}$$

$$\frac{df(\lambda)}{d\lambda} = e^{\lambda A}(A+B)\,e^{\lambda B} = e^{\lambda(A+B)}(A+B)$$

$$+\ 2\lambda t[A,B]\ e$$

$$\frac{df(\lambda)}{d\lambda} = e^{\lambda A}\,e^{\lambda B}\cdot e^{-\lambda(A+B)}$$

$$= e^{\lambda A}(A+B)\,e^{\lambda B}\,e^{-\lambda(A+B)}$$

$$+\ e^{\lambda A}\,e^{\lambda B}(-A+B)\,e^{-\lambda[A,B]}$$

$$A+\lambda[B,A]\,e^{\lambda B}$$

$$= A\,e^{\lambda A}\,e^{\lambda B}\,e^{-\lambda(A+B)} + e^{\lambda A}\left(e^{\lambda B} - A\right)e^{-\lambda[A,B]}$$

$$\Rightarrow\quad e^{\lambda B}\cdot A\,e^{-\lambda B} = A + \lambda[B,A]$$

$$= \lambda[A,B]\,f(\lambda) \Rightarrow \quad \Longrightarrow \quad \frac{1}{2}$$

Lemma 1: $e^{A} e^{B} = e^{A+B} e^{\frac{1}{2}[A,B]}$

$$[\vec{k}\cdot\vec{a}, \ \vec{k}\cdot\vec{b}] = [k_i a_i, \ k_j b_j]$$

$$= a_i b_j [k_i, k_j] = a_i b_j \, i \frac{1}{l_B^2} \epsilon_{ijk} B_k$$

$$= i(\vec{a}\times\vec{b})\cdot\vec{B}/l_B^2$$

$$k_i = -i\nabla_i - \frac{e}{c}A_i$$

$$[k_i, k_j] = [-i\nabla_i - \frac{e}{\hbar c}A_i, \ -i\nabla_j - \frac{e}{\hbar c}A_j]$$

$$= i\frac{e}{c}(\nabla_i A_j - \nabla_j A_i) = i\frac{e}{c}\epsilon_{ijk}B_k$$

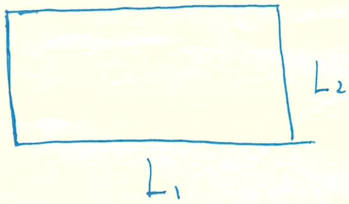
$$= i\frac{eB}{\hbar c}\cdot\cancel{\hbar} = i\frac{\cancel{\hbar}1}{l_B^2}$$

Projection operator

$$\Rightarrow \boxed{t(\hat{a})\,t(\hat{b}) = \exp\left(\frac{1}{2l_B^2}(\hat{a}\times\hat{b})\cdot\hat{z}\right) t(\hat{a}+\hat{b})}$$

$\boxed{\text{cocycle}}$

- Rectangle shape:



$$\Phi = B \cdot L_1 \cdot L_2$$

$$N_\phi = \frac{B \cdot L_1 \cdot L_2\, e}{hc} = \frac{1}{2\pi} \frac{L_1 L_2}{l_0^2}$$

$$\Phi = B \cdot l_1 \cdot L_2 \quad \cancel{}$$

$$\Rightarrow \quad \frac{L_1 \cdot L_2}{l_0^2} = 2\pi N_\phi$$

$$T_1 T_2 = T_2 T_1 \exp\left(-i \frac{L_1}{N_\phi} \hat{e}_1 \times \frac{L_2}{N_\phi} \hat{e}_2 \cdot \frac{B}{l_0^2}\right)$$

$$\Rightarrow \quad \boxed{T_1 T_2 = T_2 T_1 \exp\left(-i \frac{2\pi}{N_\phi} p\right)}$$

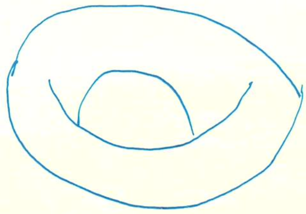
- n-th Landau level:

$$\begin{cases} H \Psi_{n,0} = E_n \Psi_{n,0} \\ T_1 \Psi_{n,0} = e^{i\lambda_0} \Psi_{n,0} \end{cases}$$

然后 $T_2^m \Psi_{n,0} = \Psi_{n,m}$ 也是本征态、

$$T_1 \Psi_{n,m} = e^{i\left(\frac{2\pi m}{N_\phi} + \lambda_0\right)} \Psi_{n,m}$$

• periodic boundary condition



$$\Psi(x_1, x_2) \equiv \Psi(x_1 + L_1, x_2) \equiv \Psi(x_1, x_2 + L_2)$$

若 N_ϕ is Quantized, N_ϕ folds - degeneracy

if N_ϕ is rational Number, functions are

Multivalued.

$$\begin{cases} P_1 = -i\nabla_1 + \frac{e}{2c} By \\ P_2 = -i\nabla_2 - \frac{e}{2c} Bx \end{cases}$$

$$P_1 + iP_2 = -i(\nabla_1 + i\nabla_2) - \frac{eiB}{2c}(x - iy)$$

$$= -i\partial_{\bar{z}} + -i\frac{eB}{2c}\bar{z}$$

~~Yang~~ - Mills - gauge field

(1)

$$\vartheta(u|\tau) = i\sum_{n=-\infty}^{n=\infty}(-1)^n \exp(i\pi\tau(n-\tfrac{1}{2})^2 + i\pi(2n-1)u)$$

$$\vartheta(u+1|\tau) = i\sum_{n=-\infty}^{n=\infty}(-1)^n \exp(i\pi\tau(n-\tfrac{1}{2})^2 + i\pi(2n-1)(u+\tau))$$

$$= i\sum_{n=-\infty}^{n=\infty}(-1) \quad \text{——} \quad = (-1)\,\vartheta(u|\tau)$$

$$\vartheta(u+i\tau|\tau) = i\sum(-1)^n \exp(i\pi\tau(n+\tfrac{1}{2}) + -$$

$-i\pi\tau$

$$i\pi(2n-1)\cdot i\tau = -\pi(2n-1)\tau$$

$$i\pi\tau(n+\tfrac{1}{2})^2 + i\pi(2n+1)u -$$

$$(-1)\,e^{i\pi}$$

- $$\vartheta(z;\tau) = \sum_{n=-\infty}^{n=\infty}\exp(i\pi n^2\tau + 2i\pi n z)$$

$$\vartheta(z+1,\tau) = \vartheta(z,\tau)$$

$$\vartheta(z+\tau;\tau) = \sum_{n=-\infty}^{n=\infty}\exp(i\pi(n+1)^2\tau + 2i\pi(n+1)z)$$

- $$\underline{\exp(-i\pi(\tau+2z))}$$

$\Downarrow$

2

- $f(z) = e^{ikz} \prod_{j=1}^{N_\phi} \vartheta_1\left(\frac{z-z_j}{L_1}\middle|\tau\right)$

$$\vartheta_1\left(\frac{z-z_j'}{L_1}\middle|\tau\right) = (i)\sum_{j=\infty}^{n=+\infty} (-1)^n \exp\left(i\pi\tau\left(n-\tfrac{1}{2}\right)^2 + i\pi(2n-1)u\right)$$

$$\vartheta_1\left(\frac{z-z_1}{L_1}\middle|\tau\right) = (i)\sum_{j=-\infty}^{n=+\infty} (-1)^n \exp\left(i\pi\tau\left(n-\tfrac{1}{2}\right)^2 + i(2n-1)\pi(u+1)\right) = (-1)\,\vartheta_1\left(\frac{z-z_1}{L_1}\middle|\tau\right)$$

$$\vartheta_1\left(\frac{z-z_j'}{L_1}+\tau\middle|\tau\right) = (i)\sum_{j=-\infty}^{n=\infty} (-1)^n \exp\left(i\pi\tau\left(n-\tfrac{1}{2}\right)^2 + i\pi(2n-1)u\right) = (i)\sum_{n=-\infty}^{n=-\infty} (-1)^n \exp\left(i\pi\tau\left(n-\tfrac{1}{2}\right)^2 + i\pi(2n-1)(u+\tau)\right)$$

$$= -e^{-i\pi\tau}\,(i)\sum_{n=-\infty}^{n=-\infty} (-1)^{n+1} \exp\left(i\pi\tau\left(n+\tfrac{1}{2}\right)^2 + i\pi(2n+1)u\right)\cdot e^{-i\pi 2u}$$

$$= (-1)\,e^{-i\pi(\tau+2u)}\,\vartheta\left(\frac{z-z_j}{L_1}\middle|\tau\right)$$

③

$$f(z+L_1) = \underbrace{(-1)^{N\phi} e^{ikL_1}}_{e^{i\theta_1}} f(z)$$

$$f(z+iL_2) = \underbrace{(-1)^{N\phi} e^{-i\pi\left(N\phi\tau + 2\sum_{j=1}^{N\phi}\frac{z-z_j}{L_1}\right)} e^{-kL_2}}$$

$$\frac{2z}{L_1}$$

$$= e^{i\theta_2} e^{-i\pi N\phi\left(\frac{2z}{L_1}+\tau\right)}$$

$$\Rightarrow \begin{cases} e^{i\theta_2} = (-1)^{N\phi} e^{i\pi\sum_{j=1}^{N\phi}\frac{2z_j}{L_1}} e^{-kL_2} \\ e^{i\theta_1} = e^{ikL_1}(-1)^{N\phi} \end{cases}$$

$$\Rightarrow \theta_1 = kL_1 \pm N\phi\pi$$

$$\theta_2 = \pi\sum_{j=1}^{N\phi}\frac{2z_j}{L_1} - ikL_2 \pm i\pi N\phi$$

- Haldane-Rezayi wavefunction

$$\boxed{\Psi_N = N\,\Psi_{cm}(z)\prod_{i<j} f(z_i-z_j)\exp\left(-\sum_{j=1}^{N}\frac{x_j^2}{2l_0^2}\right)}$$

Quantum geometry on torus:

$$-i\left(-i\partial_y+\frac{e}{\hbar c}Bx\right)$$

$$\underbrace{\phantom{-i\partial_y+\frac{e}{\hbar c}Bx}}_{p+\frac{e}{c}\vec{A}}$$

• 我们首先定义清楚 magnetic translational 算符

$$T_{lx}=e^{-ip_x\cdot l_x}=e^{-l_x\partial_x}$$

$$T_{ly}=e^{-il_y\left(p_y+\frac{e}{c}Bx\right)}=e^{-l_y\partial_y-i\frac{l_yx}{l_B^2}}$$

$$T_x\Psi(x,y)=\Psi(x-l_x,y)=e^{-ik_xx}\Psi(x,y)$$

$$\Rightarrow\quad\boxed{f(z+l_x)=e^{il_xk_x}f(z)}$$

条件1

• 现在我们考虑另外一个条件

$$T_{l_y}\Psi_{\vec{k}}(x,y) \equiv e^{-\frac{i l_y x}{l_B^2}}\Psi(x, y-l_y) \equiv e^{-ik_y l_y}\Psi(x,y)$$

$$\Rightarrow f(z-il_y)\, e^{-\frac{(y-l_y)^2}{2l_B^2}}\, e^{-\frac{i l_y x}{l_B^2}}$$

$$= e^{-ik_y\cdot l_y}\, f(z)\, e^{-\frac{y^2}{2l_B^2}}$$

$$\Rightarrow f(z) = e^{i\left(k_y\cdot l_y + \frac{l_y x}{l_B^2}\right)}\, e^{\frac{1}{2l_B^2}\left(-l_y^2 + 2l_y y\right)}$$

$$= e^{ik_y\cdot l_y}\cdot e^{i\frac{l_y}{2l_B^2}(x-iy)}\ \circ$$

<u>Where</u> : $N_\phi = \dfrac{B\cdot L_x\cdot L_y}{\frac{hc}{e}} = \dfrac{l_x l_y}{2\pi l_B^2}$

$i\pi N_\phi (z+\tau)$

$$\Rightarrow f(z+l_y) = e^{ik_y\cdot l_y}\, e^{-i\pi N_\phi(2z+\tau)}$$

考虑. First odd jacobi elliptic func

$$\vartheta(u|\tau) = i\sum_{n=-\infty}^{n=\infty}(-1)^n \exp(i\pi(n-\tfrac{1}{2})^2\tau + i(2n-1)\pi u)$$

Let: $q = e^{i\pi\tau} = e^{-i\pi\frac{dy}{dx}}$

$$(2n-1)+(2k-1)=0 \Rightarrow n+k=1$$

$$\Rightarrow \vartheta(u|\tau) = i\sum_{n=-\infty}^{n=\infty}(-1)^n q^{(n\mp 1/2)^2}\sin(2n\mp 1)\pi u$$

$$(-1)^n q^{(n-\frac{1}{2})\tau}\Big(\underbrace{e^{i\pi u(2n-1)} - e^{-i\pi u(2n-1)}}_{(2i)}\Big)$$

$$-(n\mp 1)-\tfrac{1}{2}$$

$$\vartheta(u|\tau) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\sin(2n+1)u \times (-1)^n$$

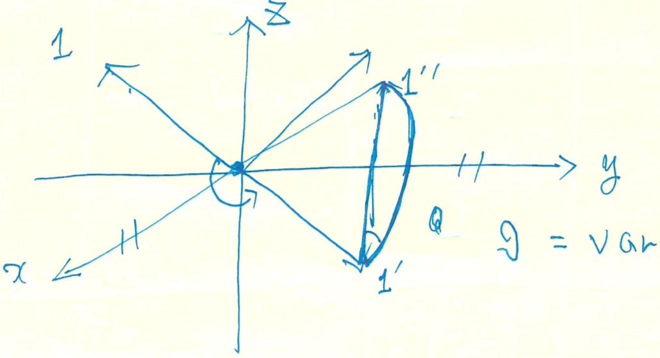
$$\vartheta(u+1|\tau) = -\vartheta(u|\tau)$$

$$\vartheta(u+\tau|\tau) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\sin(2n+1)(u+\tau)$$

$$\exp(i\pi\,\hat{n}_1\cdot\vec{\sigma})\cdot\exp(i\,\hat{n}_2\cdot\vec{\sigma})$$

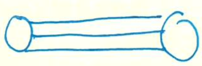
$$= \exp(i\pi(\hat{n}_1+\hat{n}_2)\cdot\vec{\sigma})$$

two half turn:



$$\cancel{\cos\frac{\pi}{2}}\quad i\sin\left(\frac{\pi}{2}\right)(\hat{n}_1\cdot\vec{\sigma}_1) = i(\hat{n}_1\cdot\vec{\sigma}_1)\cdot(\hat{n}_2\cdot\vec{\sigma}_2)$$

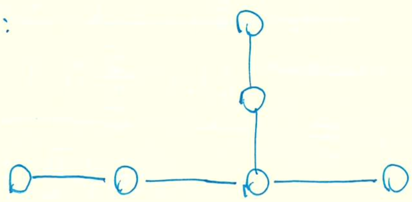
$$= i\left((\hat{n}_1\cdot\hat{n}_2) + i(\hat{n}_1\times\hat{n}_2)\cdot\vec{\sigma}\right)$$



G_2

F_4 F_4

$E_{6,7,8}$:



简单 Lie algebra